

Recovering Deflections of Vertical from A Tangent Plane of Gridded Geoidal Heights from Altimetry

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Abstract. To improve the space resolutions of vertical deflections and gravity anomaly recovering from altimetry data, a new method for computing vertical deflections is presented. At first, we eliminate the influence of sea surface topography from the mean sea surface height of altimetry data along ground tracks, and regard the results as geoidal heights. The data coordinates are then transformed into new Cartesian coordinates, which consist of a osculating tangent plane and a normal of the reference ellipsoid. Then, we fit a osculating tangent plane of the local geoid using a least-squares approach by minimizing the squared summation of the total distance between the discrete points and the fitting plane; this is to determinate the mean value of each component of the vertical deflections. An experiment was made in the area of the South China Sea. The comparisons between the vertical deflections determined by this method and those by other methods show that the precision of this method agrees well with those from other methods on 5'×5' grid.

Keywords. Satellite altimetry, deflections of the vertical, osculating tangent plane, least squares approach.

1 Introduction

Along with the development of altimetry technology, much altimetry satellite data was used in working out a high resolution of oceanic gravity anomaly. For example, Sandwell et al. (1997) and Hwang et al. (1998) had computed global gravity anomaly by using both Geosat/GM and ERS1 altimetry data on a 2'×2' grid, Wang et al. (2000) had determined gravity anomaly in the South China Sea by combining Geosat/GM, ERS1 and T/P altimetry data on a 2'×2' grid, and Li et al. (2001) determined gravity anomaly in the Chinese seas and its vicinity by using Geosat/GM, ERS2 and T/P altimetry data on a 2.5'×2.5' grid. They all adopt the method of computing gravity anomaly from the

vertical deflections at the crossovers of satellite tracks or along satellite tracks, which is the so-called “vertical deflections method”. According to the comparisons, this approach is the best method in computing a higher resolution gravity anomaly at present, see Huang et al. (2001). Nevertheless, because the vertical deflections are localized at the crossovers points of satellite tracks and the spatial distribution of crossover points is limited, it is difficult to recover gravity anomaly from altimetry data with a resolution higher than 2'. Therefore, it would be important to improve the arithmetic of computing vertical deflections. At present, the methods used in computing vertical deflections are mainly the method for computing the first difference of mean sea surface height (or geoidal height) (Anzenhofer et al., 1998, Cazenave et al., 1996), the method for computing the time differential coefficient along satellite tracks, the method computing the vector product of ascend arc, and descend arc of tracks (Kim et al., 1996).

In this paper, we present a “geoid quasi tangent plane method (GQTPM)” to compute vertical deflections, which uses all valid altimetry data in order to increase the resolution of vertical deflections and recover a high-resolution gravity anomaly.

2 Theory and Method of the GQTPM

Firstly, a block is selected (the block size is equal to the resolution of the result). Each block is formed by curvilinear coordinates consisting of longitude, latitude and the outer normal of a reference ellipsoid. The origin of the curvilinear coordinates is at the lower left corner of the block. The arc-length along latitude from a data point to the origin O is defined as u_1 , the arc-length along longitude from a data point to origin O is defined as u_2 , and the geoidal height is defined as u_3 .

The original altimetry data are geocoded in geodetic coordinates, which are transformed to Cartesian coordinates, see Fig.1. A tangential plane

is constructed which osculates the reference ellipsoid at the lower left corner of block. In this tangential plane, the projection of the longitude circle is regarded as the X-axis, the projection of the latitude circle as the Y-axis, and the outer normal of the reference ellipsoid as the Z-axis.

In this Cartesian coordinates, the X-axis is the projection of u_1 along the outer normal, the Y-axis is the projection of u_2 along the outer normal, and the Z-axis holds the line.

The approximate conversion equation of two coordinates is,

$$\begin{cases} x = \pi \cdot R_0 \cos(\varphi_0) \cdot \frac{(\lambda - \lambda_0)}{180} \\ y = \pi \cdot R_0 \cdot \frac{(\varphi - \varphi_0)}{180} \\ z = h \end{cases} \quad (1)$$

Where, R_0 is the distance between the origin and the geocenter, $(\varphi_0, \lambda_0, h)$ are geodetic coordinates, and (φ, λ) are the latitude and longitude of the observation point.

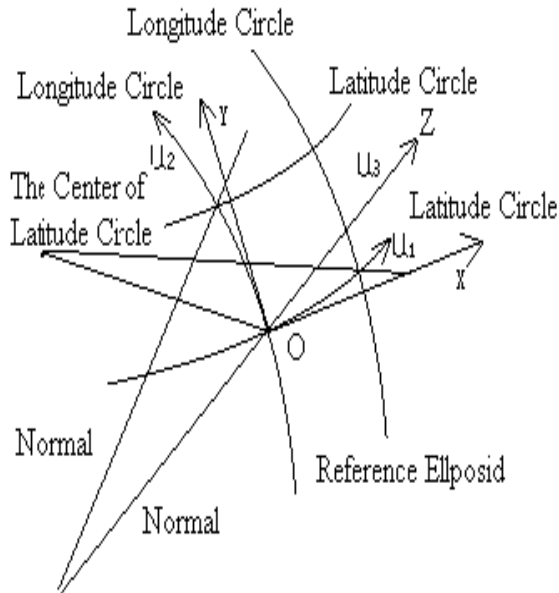


Fig.1 Transform between the curvilinear coordinates and Cartesian coordinates.

In the Cartesian coordinates, the common equation of a plane is $Ax + By + Cz + D = 0$, where A, B, C, D are constants. Generally speaking, D is not

equal to zero and we can simplify this equation to $Ax + By + Cz + 1 = 0$.

The basic idea of this method is to use the osculating tangential plane to fit the geoidal heights in a given block, and then to denote the inner normal direction of the tangential plane as the inner normal direction of the local geoid. As such, we can derive the two components of vertical deflections as $\arctan(-A/C)$ and $\arctan(-B/C)$.

Let d_n be the vertical distance between the n th data point and the tangential plane. By this definition, d_n is the difference of tangential plane and the local geoid. We can measure the degree of closeness between the data points and the tangent plane by $\sum_{n=1}^N d_n^2$.

According to the analytic geometry, the vertical distance between a data point and the fitting plane is

$$d_n = \frac{(Ax + By + Cz + 1)}{\sqrt{A^2 + B^2 + C^2}}. \quad (2)$$

We have $\sum_{n=1}^N d_n^2 = \sum_{n=1}^N \frac{(Ax + By + Cz + 1)^2}{A^2 + B^2 + C^2}$, where N is the total number of the discrete points.

We can denote $\sum_{n=1}^N d_n^2$ by $f(A, B, C)$. For convenience the notation $\sum_{n=1}^N$ is simplified as $\sum$.

It is assumed that, when N is sufficiently large, only one fitting plane satisfies the minimizing conditions, leading to

$$f'_A(A, B, C) = 0, f'_B(A, B, C) = 0, f'_C(A, B, C) = 0$$

where f' is the differential to f .

With simplification we obtain

$$\begin{cases} B \sum (Ax_n + By_n + Cz_n + 1) \cdot x_n = A \sum (Ax_n + By_n + Cz_n + 1) \cdot y_n \\ C \sum (Ax_n + By_n + Cz_n + 1) \cdot x_n = A \sum (Ax_n + By_n + Cz_n + 1) \cdot z_n \\ C \sum (Ax_n + By_n + Cz_n + 1) \cdot y_n = B \sum (Ax_n + By_n + Cz_n + 1) \cdot z_n \end{cases} \quad (3)$$

An additional restrictive condition is needed in order to solve the equations. In general, the tangential plane of the local geoid could not be perpendicular to the reference ellipsoid and hence $C \neq 0$. For any plane $Ax + By + Cz + D = 0$ with $D \neq 0$, a necessary condition to have an extremum of

$\sum d_n^2$ is $f'_D(A, B, C, D) = 0$, which means

$$f'_D(A, B, C, D) = \frac{\partial \left(\sum \frac{(Ax + By + Cz + D)^2}{A^2 + B^2 + C^2} \right)}{\partial D} = 0 \quad (4)$$

That is, $\sum (Ax + By + Cz + D) = 0$. Therefore, we have

$$A \sum x_n + B \sum y_n + C \sum z_n + N = 0 \quad (5)$$

and

$$A \frac{\sum x_n}{N} + B \frac{\sum y_n}{N} + C \frac{\sum z_n}{N} + 1 = 0 \quad (6)$$

These formulas show that the geometric center of discrete points, $\left(\frac{\sum x_n}{N}, \frac{\sum y_n}{N}, \frac{\sum z_n}{N} \right)$, should be in the fitting plane.

For the convenience, we change coordinates from the old coordinates to a new coordinates, whose origin point is at $\left(\frac{\sum x_n}{N}, \frac{\sum y_n}{N}, \frac{\sum z_n}{N} - 1 \right)$. In this new coordinates, the coordinates of the discrete points are denoted by (x', y', z') , see Fig.2.

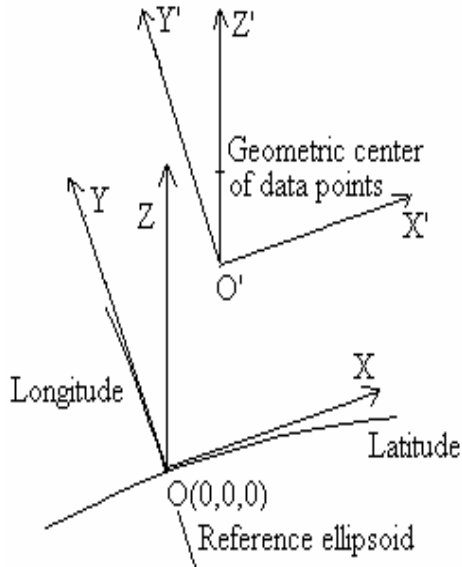


Fig.2 Transform of Coordinates

With previous statements, we have already made a restrictive condition (eq. (4)), which means that the fitting plane will pass the geometric center of

discrete points. This is true for the new coordinates, i.e. the fitting plane will also pass the new coordinate of geometric center $(0, 0, 1)$. In the new coordinates, the equation of the fitting plane becomes $A'x'_n + B'y'_n + C'z'_n + 1 = 0$. This fitting plane are for the new and old coordinates, so we can use (A', B', C') to express the inner normal direction of the fitting plane. Coordinate transformation is expressed as

$$\begin{cases} x'_n = x_n - \frac{\sum x_n}{N} \\ y'_n = y_n - \frac{\sum y_n}{N} \\ z'_n = z_n - \frac{\sum z_n}{N} + 1 \end{cases} \quad (7)$$

with

$$\sum x'_n = 0 \quad \sum y'_n = 0 \quad \sum z'_n = N. \quad (8)$$

In the $o' - x'y'z'$ Cartesian coordinate system, the equations in (2) still hold. Substituting the expressions of x', y', z' into these equations leads to eq. (9).

This is a linear-correlative equation containing dual-variants with a maximal exponent other than one. Removing $A'B'$ and B'^2 terms, we get eq. (10), which just contains B', A' and A'^2 terms.

If the coefficient of B' term is equal to zero, eq. (10) will become an equation containing one variant with a maximal exponent being square. We can obtain the values of A' directly according to known formulas, and then apply the value of A' to the second line of eq. (10). Finally, the values of B' can be obtained. If the coefficient of B' term is unequal to zero, we may put the dependence relationship between B' and A', A'^2 into the second formula of eq. (10) and obtain an equation containing one variant A' with a maximal exponent being cubic. Because any equation having one variant and a maximal exponent being cubic can be transformed into the form $x^3 + px + q = 0$, we can get all possible results of A' according to a known Cardan formula, then apply the values of A' back to the functional relationship between B' and A', A'^2 , and finally obtain the values of B' . Considering numerical accuracy of computer, the results of A' and B' obtained with above steps are just approximations. Regarding the values of A', B' as

$$\begin{aligned}
& [(\sum x'_n y'_n)^2 \cdot \sum (x'_n z'_n) - \sum (x'_n y'_n) \cdot \sum (y'_n z'_n) \cdot (N + \sum (x_n'^2 - z_n'^2)) - (\sum y'_n z'_n)^2 \cdot \sum (x'_n z'_n)] \cdot B' \\
& + [(\sum x'_n z_n'^2 \cdot \sum (x'_n y'_n) - (\sum y'_n z_n'^2 \cdot \sum (x'_n y'_n) - \sum (x'_n y'_n) \cdot \sum (y'_n z'_n) \cdot \sum (x_n'^2 - y_n'^2))] \cdot A'^2 \\
& + [\sum (x'_n y'_n) \cdot \sum (x'_n z'_n) \cdot (N + \sum (x_n'^2 - z_n'^2)) - (\sum x'_n y'_n)^2 \cdot \sum (y'_n z'_n) - (\sum y'_n z'_n)^3 \\
& - \sum (y'_n z'_n) \cdot \sum (x_n'^2 - y_n'^2) \cdot (N + \sum (x_n'^2 - z_n'^2))] \cdot A' \\
& + [\sum (x'_n y'_n) \cdot (\sum y'_n z'_n)^2 - \sum (x'_n y'_n) \cdot (\sum x'_n z'_n)^2 + \sum (x'_n z'_n) \cdot (\sum y'_n z'_n) \cdot \sum (x_n'^2 - y_n'^2)] = 0
\end{aligned} \tag{9}$$

$$\begin{cases}
\sum (y'_n z'_n) \cdot A' - \sum (x'_n z'_n) \cdot B' - \sum (x'_n y'_n) \cdot A'^2 + \sum (x'_n y'_n) \cdot B'^2 + \sum (x_n'^2 - y_n'^2) \cdot A' B' = 0 \\
[-N - \sum (x_n'^2 - z_n'^2)] \cdot A' - \sum (x'_n y'_n) \cdot B' - \sum (x'_n z'_n) \cdot A'^2 - \sum (y'_n z'_n) \cdot A' B' + \sum x'_n z'_n = 0 \\
-\sum (x'_n y'_n) \cdot A' + [-N - \sum (y_n'^2 - z_n'^2)] \cdot B' - \sum (y'_n z'_n) \cdot B'^2 - \sum (x'_n z'_n) \cdot A' B' + \sum y'_n z'_n = 0
\end{cases} \tag{10}$$

the initial values of Newton-iteration, we will obtain the final values of A' and B' with a sufficient precision. If there is only one A' value, what we need to do is just put it into the relationship equation between B' and A' , A'^2 , and obtain the values of B' . If there is more than one A' value, we should apply the values of A' and B' to $\sum d_n'^2$ and select the proper values of A' and B' minimizing $\sum d_n'^2$, and these are the final values. Thus, we have obtained the orientation of the inner normal of the fitting plane (A' , B' , -1). Because two components of vertical deflections are defined on the westward and southward directions, we must transform the values of A' , B' into these two directions employing the following formula.

$$\begin{aligned}
\eta &= -206265 \cdot \arctan(A') \\
\xi &= -206265 \cdot \arctan(B')
\end{aligned} \tag{11}$$

where η and ξ are the eastern and western components of vertical deflections, respectively. The units are arc-second.

3 Experiment and Comparisons

To validate this method, we made an experiment in the South China Sea. Here we computed geoid height in the area 0-25°N, 105-122°E with 3' grid from the regional geopotential model IGG_SCS00A (Yang, 2002) complete to degree and order 3600, named, and interpolated it into a 1'×1' grid, which will be regarded as simulated observations. In total, there are 1,091,913 values. Then, the two components of vertical deflections on a 5'×5' gridded computed from this model are regarded as standard values, which will be used for calibration and validation of the result computed by the

method of GQTPM.

Based on GQTPM, high-resolution vertical deflections on a 5'×5' grid are computed in the South China Sea, with a total of 43,829 values. The statistics of the results by this method are summarized in Table 1.

Table1. The statistics of deflections of vertical, in arc-second

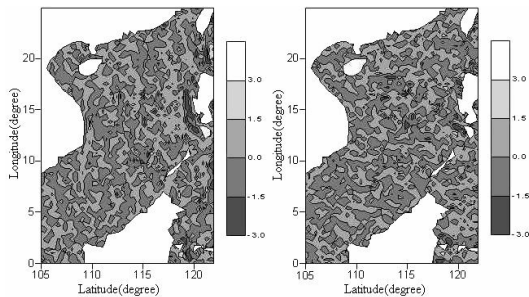
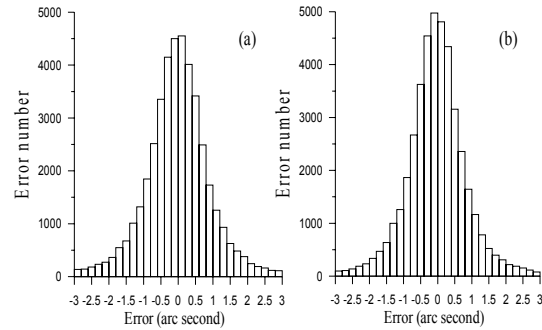
Item	η	ξ
Data Number	43829	43829
Min. (arc sec)	-41.6090	-28.8108
Max. (arc sec)	65.6915	43.9959
Mean (arc sec)	-6.7536	3.9539
Std. dev.(arc sec)	5.0870	4.4732

To validate our results, we regard the parallels and the meridians on the 5'×5' grid as ascending tracks and descending tracks, and the grid points as the crossover points of the tracks. For comparison, we computed various values of vertical deflections by different methods, i.e. the GQTPM method, the first difference of geoidal height method using adjacent grid data (Anzenhofer et al., 1998; Cazenave et al., 1996) (for short as 'D' method), the Watts vector product method (Watts, 1984) (for short as 'V' method), and the IGG_SCS00A model (for short as 'M' method). Here, the adjacent grid points are used in the first difference of geoidal height method, parallels and meridians are used as ascending and descending tracks in order to obtain vector product of two tracks. We eliminate those data whose absolute differences with others values exceed $\pm 3''$. The statistics of the comparisons are shown in Table 2.

Table 2. The statistics of results from comparing different methods, in arc-second

	M- GQTPM		M-D		M-V		V- GQTPM		D- GQTPM	
	η	ξ	η	ξ	η	ξ	η	ξ	η	ξ
$>\pm 3''$	3.02%	2.08%	2.74%	2.98%	2.77%	1.81%	0.11%	0.17%	0.38%	0.91%
Num.	41960	42369	38460	38569	42076	42427	42893	42859	43403	43404
Max	2.9978	2.9969	2.9993	2.9968	-2.9944	-2.9968	2.9802	2.9866	2.9952	2.9980
Min	-2.9998	-2.9996	-2.9977	-2.9973	2.9990	2.9874	-2.9959	-2.9669	-2.9998	-2.9951
Mean	-0.0205	-0.034	-0.0582	0.2775	-0.0269	0.0609	0.0014	-0.1143	0.0358	-0.3400
Std dev	0.8913	0.8410	0.8549	0.8994	0.8633	0.8226	0.5054	0.5190	0.6014	0.6725
RMS	0.8916	0.8417	0.8569	0.9412	0.8637	0.8248	0.5054	0.5314	0.6025	0.7536

According to the comparisons in Table 2, we can see that in the research area the errors of GQTPM method in the east-western and south-northern directions are $\pm 0.89''$ and $\pm 0.84''$, the errors of the first difference of geoidal height method are $\pm 0.86''$ and $\pm 0.94''$, the errors of the vector product method are $\pm 0.86''$ and $\pm 0.82''$, respectively. By this token, the precisions of the GQTPM methods are about $\pm 0.9''$. In other words, the precision of the GQTPM method is similar to those from other methods, so the GQTPM method is feasible and accurate. Fig.3 illustrates errors distribution of two components of vertical deflections. From these two pictures, it is obvious that errors distributes discretely. Fig.4 shows the errors histogram of two components of vertical deflections. From Fig. 4, most of them distribute between $-1.5''$ and $1.5''$, which are consistent with the rule of error distribution and shows the validity of the GQTPM method again.

**Fig. 3** The distributing contour map of difference of η , ξ between deflections of vertical computed from IGG_SCS00A model and the GQTPM method.**Fig. 4** Histogram of errors between IGG_SCS00A model and the GQTPM method. (a) Histogram of east-west component of vertical deflections. (b) Histogram of south-north component of vertical deflections.

4 Conclusions

During the past decades, there are plenty of altimeter missions providing abundant and valuable data for the geophysical, geodetic and oceanography applications. Because accurate and detailed vertical deflections over the ocean are basic data for various applications, it has attracted more and more attention. The gridded values of vertical deflections determined by the method of GQTPM, as a whole, reflect correctly the distribution of vertical deflections in the research area. Comparisons with other methods show that it is feasible and accurate for computing vertical deflections.

In the raw satellite altimeter data, ground tracks of different cycles do not repeated entirely, there exists a distance between each ground track and the mean track of all tracks. Therefore, in fact, the data points are distributed in a zone along ground tracks. When we combine multi-satellite altimeter data, in the region near mean tracks of all satellite's ground tracks, data will be distributed discretely and will

exceed the total number of the crossover points and normal points. With GQTPM, we can compute vertical deflections directly from the discrete data, without co-line computation for informing linear track. Theoretically, we can determine vertical deflections not only at the crossover points and normal points but also at the region between crossover points using GQTPM. As a result, it can be used to recover vertical deflections and from altimetry data gravity anomalies with a higher resolution.

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